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DIAMETER VULNERABILITY OF GRAPHS

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Let $f(t,D)$ denote the maximum possible diameter of a graph obtained from a $(t+1)$-edge-connected graph of diameter D by deleting t edges. F.R.K. Chung and M.R. Garey have shown that for $D\geq 4$, $(t+1)(D-2)\leq f(t,D)\leq (t+1)D+t$. Here we consider the cases $D=2$ and $D=3$ and show that $f(t,2)=4$ and $3\sqrt{2t}-3\leq f(t,3)\leq 3\sqrt{2t}+4$ if t is large enough. We solve also the problem for the directed case (answering a question of F.R.K. Chung and M.R. Garey) by showing that if $D\geq 3$ the diameter of a digraph obtained from a $(t+1)$-arc-connected digraph of order n by deleting t arcs is at most $n-2t+1$. In the case $D=2$, the maximum possible diameter of the resulting digraph is (like in the undirected case) 4. We also consider the same problem for vertices.

1. Introduction

Let $G=((V(G),E(G))$ be a directed or undirected graph of order n with vertex-set $V(G)$ and edge-set $E(G)$. The distance $d_G(x,y)$ from x to y is the length of a shortest path from x to y. The diameter $D(G)$ of G is the maximum value of $d_G(x,y)$ over all pairs of vertices x,y in $V(G)$. A graph (a digraph) has finite diameter if and only if it is connected (strongly connected). If x is any vertex in $V(G)$, $\Gamma(x)$(resp. $\Gamma^+(x),\Gamma^-(x)$) is the set of all neighbours (resp. successors, predecessors) of x, and the degree $d(x)$ (resp. outdegree $d^+(x)$, indegree $d^-(x)$) of x is equal to the cardinality of $\Gamma(x)$ (resp. $\Gamma^+(x),\Gamma^-(x)$).

Let $f(t,D)$ denote the maximum possible diameter of a graph obtained from a $(t+1)$-edge-connected graph by deleting t edges. This notion has been introduced by F.R.K. Chung and M.R. Garey in [1], to which we refer the reader for references on analogous problems. They showed that for fixed t and $D\geq 4$

$$(t+1)(D-2)\leq f(t,D)\leq (t+1)D+t.$$

Here we consider the cases $D=2$ and $D=3$, and show that $f(t,2)=4$ and $3\sqrt{2t}-3\leq f(t,3)\leq 3\sqrt{2t}+4$. At the end of their article they set the same problem for directed graphs. Here we show that the situation is completely different for $D\geq 3$: if G is a $(t+1)$-arc-connected digraph of order n and diameter at least 3, the maximum possible diameter for a digraph obtained from G by deleting t arcs is $n-2t+1$, and this bound can be achieved. In the case of diameter 2 the maximum possible diameter is 4. We also consider the same problem for deletion of vertices.

2. Case $D=2$

Theorem. $f(t, 2)=4$.

Proof. We will first prove that $f(t, 2) \leq 4$. That will follow from the following lemma (the idea of its proof is due to P. Fraisse).

Lemma. *Let G be a graph of diameter 2 and G' the graph obtained from G by deleting strictly less than δ edges (where δ is the minimum degree of G). Then the diameter of G' is at most 4.*

Proof. We will prove in fact that between any pair of vertices x, y of $V(G)$, there are at least $\min(d(x), d(y))$ edge-disjoint paths of length at most 4.

Without loss of generality, let us suppose that $d(x) \leq d(y)$ and let $\Gamma(x) \cap \Gamma(y) = A=\{a_1, \ldots, a_h\}$; $\Gamma(x)-A-y=X=\{x_1, \ldots, x_u\}$; $\Gamma(y)-A-x=\{y_1, \ldots, y_{u'}\}$. We have $u \leq u'$ and $d(x)=h+u+e$, where $e=1$ if x and y are joined by an edge and 0 otherwise.

As $d(x_i, y_i) \leq 2$, we have a path P_i between x_i and y_i of length at most 2; that is P_i is either the edge $\{x_i, y_i\}$ or the path x_i, b_i, y_i where $b_i \neq x, y$.

Then consider the $d(x)$ following paths:

- the edge $\{x, y\}$ if it exists,
- the h paths $Q_i=\{x, a_i, y\}$ where $a_i \in A$,
- the u paths $\{x, P_i, y\}$.

These paths are edge-disjoint and of length at most 4. □

Let us show that $f(t, 2)=4$. In fact there exist graphs of arbitrary high edge-connectivity of diameter 2 such that if we delete some particular edge the graph obtained has diameter 4.

Let $V(G)=\{x\} \cup\{y\} \cup\{X\} \cup\{Y\} \cup\{Z\}$ and the edges be the edge $\{x, y\}$, all the edges between x and X, all the edges between y and Y and all the edges joining any vertex of Z to any vertex of X or Y.

If $|X|=|Y|=|Z|=\delta-1$, G is of minimum degree δ, diameter 2 and $G-\{x, y\}$ is of diameter 4. □

3. Case $D=3$

Theorem. $3\sqrt{2t}-3 \leq f(t, 3) \leq 3\sqrt{2t}+4$ *if t is large enough.*

Proof. (I) We will first prove that $f(t, 3) \leq 3\sqrt{2t}+4$ if t is large enough.

Notations. Let G be a $(t+1)$-edge-connected graph of diameter 3. Let G' be the graph obtained from G by deleting a set E of t edges. G' is connected, let

D be its diameter. Let x, y be two vertices at distance D in G' and let $P = \{x = x_0, x_1, \ldots, x_i, \ldots, x_D = y)$ be a path of length D in G'. Let

$$N_i = \{z \in V(G) \mid d_{G'}(x, z) = i\},$$

$$L_i = N_i \cup \Gamma(N_i) = \{z \in V(G) \mid z \in N_i \text{ or there exists } z' \in N_i \text{ with } d_{G'}(z, z') = 1\},$$

$$e_i = \text{number of edges of } E \text{ having } x_i \text{ as endvertex.}$$

Note that N_i and L_i are not empty as they contain x_i.

(a) $$|N_0| + |N_1| \geq t + 2 - e_0, \qquad |N_{D-1}| + |N_D| \geq t + 2 - e_D,$$

$$|N_{i-1}| + |N_i| + |N_{i+1}| \geq t + 2 - e_i \quad \text{for } 1 \leq i \leq D-1.$$

Proof. Indeed $d_G(x_i) \geq t + 1$ and $d_{G'}(x_i) \geq t + 1 - e_i$, furthermore $\{x_0\} \cup \Gamma(x_0) \subset N_0 \cup N_1$, $\{x_D\} \cup \Gamma(x_D) \subset N_{D-1} \cup N_D$ and $\{x_i\} \cup \Gamma(x_i) \subset N_{i-1} \cup N_i \cup N_{i+1}$ (for $1 \leq i \leq D-1$).

(b) Let $p + 1 = \lceil (D+1)/3 \rceil$; for $0 \leq k \leq p$ let $I_k = \{3k-1, 3k, 3k+1\} \cap [0, D]$ ($I_0 = \{0, 1\}$, $I_1 = \{2, 3, 4\}, \ldots, I_p = \{3p-1, 3p\}$ if $D = 3p$ and $I_p = \{3p-1, 3p, 3p+1\}$ if $D = 3p+1$ or $3p+2\}$. Now, for $0 \leq k \leq p$ let j_k be chosen in I_k such that $|N_{jk}|$ is maximum over the $|N_i|$ where $i \in I_k$. Let $J = \{j_0, j_1, \ldots, j_k, \ldots, j_p\}$, $|J| = p + 1$.

By (a) and the maximality of $|N_{jk}|$ we have $|N_{jk}| \geq \lceil (t + 2 - e_{3k})/3 \rceil$.

(c) Let i and j be such that $|j - i| > 3$. Then $L_i \cap L_j = \emptyset$ and there is no edge in G' between L_i and L_j.

Proof. If $z \in L_i$, then $i - 1 \leq d_{G'}(x, z) \leq i + 1$.
If $t \in L_j$, then $j - 1 \leq d_{G'}(x, t) \leq j + 1$.
If $|j - i| \geq 4$, then $|d_{G'}(x, z) - d_{G'}(x, t)| \geq 2$ and therefore $d_{G'}(z, t) \geq 2$. □

(d) Let i and j be two elements of J such that $|j - i| \geq 4$. If there is no edge of E between L_i and L_j then there are at least $\lceil (t+1)/3 \rceil$ edges of E having one of their endvertices in L_i (or L_j).

Proof. Suppose that there exist $z \in N_i$ and $t \in N_j$ such that neither z nor t is the endvertex of an edge of E, with its other vertex not in L_i (L_j). Therefore all the neighbours of z (resp. t) in G are in L_i (resp. L_j). But there is no edge in G, by (c) and the hypothesis, between L_i and L_j; therefore $d_G(z, t) > 3$ contradicting the fact that the diameter of G is 3.

So, without loss of generality, we can suppose that each vertex of N_i is the endvertex of an edge of E, whose other endvertex does not belong to L_i. There exists j_k such that $i = j_k$ and, by (b), $|N_i| \geq \lceil (t + 2 - e_{3k})/3 \rceil$. If $e_{3k} = 0$, then we have $|N_i| \geq \lceil (t+2)/3 \rceil$. If $e_{3k} \neq 0$, we have the e_{3k} edges in E, which have x_{3k} as endvertex and $|N_i| - 1$ edges going out of N_i (the -1 comes from the fact that x_{3k} can belong to N_i). These edges are different from the preceding ones because their other endvertex is not in L_i but x_{3k} is in L_i. In summary we have exhibited $e_{3k}+\lceil (t+2-e_{3k})/3\rceil -1$ edges having one of their endvertices in L_i. But $e_{3k}+\lceil (t+2-e_{3k})/3\rceil -1 \geq \lceil (t+1)/3\rceil$ thus proving (d). □

(e) If $p+1=\lceil (D+1)/3\rceil \geq 13$, then $p\leq \sqrt{2t}+1$ and $D\leq 3\sqrt{2t}+4$.

Proof. Let K be the set of indices i such that at least $\lceil (t+1)/3\rceil$ edges of E have one endvertex in L_i. Therefore for i,j in $J-K$ such that $|j-i|\geq 4$ there is by (c) and (d) an edge of E between L_i and L_j.

Let $k=|K|$. We obtain

$$t=|E|\geq \frac{k}{2}\left\lceil \frac{t+1}{3}\right\rceil + \frac{(|J|-k)(|J|-k-3)}{2},$$

therefore $k\leq 5$. As $|J|=p+1$ we have

$$t\geq \frac{k}{2}\frac{(t+1)}{3}+\frac{(p+1-k)(p-2-k)}{2}, \tag{R}$$

$$t\geq \frac{3p^2-3(2k+1)p+(3k^2+4k-6)}{6-k}.$$

But as soon as $p\geq 12$ we have

$$\frac{3p^2-3(2k+1)p+(3k^2+4k-6)}{6-k}\geq \frac{(p+1)(p-2)}{2}.$$

If $t\geq (p+1)(p-2)/2$ and $p\geq 12$, then $p\leq (1+\sqrt{9+8t})/2\leq 0.6+\sqrt{2t}$. Therefore, $D\leq 3p+2\leq \sup(3\sqrt{2t}+3.8;\ 35)$, as for $t\geq 54$ $3\sqrt{2t}+4\geq 35$, we have the upper bound of the theorem. For smaller values of t we can improve the calculation.

For example, if $p=11$, then by (R) we have $t\geq 53$ and $\sqrt{2t}+1\geq \sqrt{106}+1\geq 11$ and so $p\leq \sqrt{2t}+1$.

If $p=10$, then by (R) we have $t\geq 41$ and $\sqrt{2t}+1\geq \sqrt{82}+1\geq 10$ and so $p\leq \sqrt{2t}+1$.

For $p\geq 10$ we obtain that $p\leq \sqrt{2t}+1$.

If $p\leq 9$ we have that $D\leq 3p+2\leq 29$.

Therefore, we always have $D\leq \sup(3\sqrt{2t}+5, 29)$.

If $t\geq 32$ we have $3\sqrt{2t}+5\geq 29$, and the theorem is proved.

For small values of t (≤ 31), careful calculations give a smaller bound than 29 for D.

(II) We will now prove that $f(t,3)\geq 3\sqrt{2t}-3$.

For any t there exists some p such that $p(p-1)/2\leq t<(p+1)p/2$. Let $D=3p-1$. Let G be the graph constructed as follows: the vertices of G are partitioned into $D+1$ sets $V_0, V_1, \ldots, V_D$ such that for any i, $0<i<D$:

$$|V_i|=k_1=\lceil t/3\rceil +1, \qquad |V_0|=|V_D|=t+2-k_1.$$

There is an edge between x_i in V_i and y_j in V_j if and only if $|j-i|\leq 1$. This graph has diameter D and is $(t+1)$-edge-connected. For $i\equiv 1 \pmod 3$, let us choose a vertex x_i in V_i. The number of such vertices is p. Now let us add all the possible edges between the x_i, and $t-p(p-1)/2$ other edges anywhere in the graph. The resulting graph has diameter 3 because every vertex is at distance at most one of the x_i. It is still $(t+1)$-edge-connected. By deleting the t additional edges, we obtain a graph of diameter D. This implies that $f(t,3)\geq D$, where $D=3p-1$ and $t<(p+1)p/2$. $t<p(p+1)/2$ implies that $p\geq(\sqrt{8t+1}-1)/2\geq\sqrt{2t}-\frac{1}{2}$. By consequence $f(t,3)\geq 3\sqrt{2t}-\frac{3}{2}-1\geq 3\sqrt{2t}-3$. $\square$

4. Directed case

We will show that on the contrary with the undirected case, if the diameter of the original digraph is at least 3, the diameter of the resulting digraph can be as large as we want.

Theorem. *Let G be any $(k+1)$-arc-connected digraph of diameter at least 3 and order n. Let G' be the digraph obtained from G by deleting the k arcs of a set $E\subset E(G)$. Then the diameter of G' is at most* $\sup(4, n-2k+1)$, *and this bound can be achieved.*

Proof. We will suppose that $D(G')\geq 5$.

Let x and y be two vertices of G such that $d_{G'}(x,y)=D'$ and let P be a path of length D' from x to y in G'. Then $P=(x=x_0,x_1,\dots,x_{D'-1},x_{D'}=y)$. Let $X=\Gamma^+(x)-\{x_1\}$, $Y=\Gamma^-(y)-\{x_{D'-1}\}$, $R=\Gamma^+(X)-X-\{x_0,x_1,x_2\}$, $S=\Gamma^-(Y)-Y-\{x_{D'},x_{D'-1},x_{D'-2}\}$ and $T=V(G)-\{X\cup Y\cup R\cup S\cup V(P)\}$. Note that these sets are disjoint sets.

If $|X|+|Y|+|R|+|S|+|T|\geq 2k-2$, then $D'\leq n-2k+1$.

If $|X|+|Y|+|R|+|S|+|T|<2k-2$ we can suppose that $|X|+|R|\leq k-2$ (the other case $|Y|+|S|\leq k-2$ can be dealt with in a symmetric way).

In G' we have

$$d^+_{G'}(x)=1+|X|,\qquad d^+_{G'}(x_1)\leq|X|+|R|+2\leq k$$

and for every $u\in X$, $d^+_{G'}(u)\leq|X|+|R|+2\leq k$ but as G is $(k+1)$-arc-connected, for every $v\in V(G)$ $d^+_G(v)\geq k+1$. Then we have

$$d^+_G(x)-d^+_{G'}(x)\geq k-|X|$$

and for every $v\in\Gamma^+(x)$, $d^+_G(v)-d^+_{G'}(v)\geq 1$ and

$$|E|\geq\sum_{v\in\{x\}\cup\Gamma^-(x)}(d^+_G(v)-d^+_{G'}(v))\geq k-|X|+|X|+1=k+1,$$

contradicting the fact that $|E|=k$. $\square$

This bound can be achieved: let G be the graph constructed as follows: G consists of a path of length D' whose vertices are $x_0,x_1,\dots,x_{D'}$, plus $2(k-1)$ vertices $\{(a_i,b_i)\ 1\leq i\leq k-1\}$, with the following additional arcs:

$$
\begin{array}{llll}
(x_0, x_{D'}), & & & \\
(a_i, a_j), & 1 \le i \le k-1, & 1 \le j \le k-1, & i \ne j, \\
(b_i, b_j), & 1 \le i \le k-1, & 1 \le j \le k-1, & i \ne j, \\
(a_i, b_i), & 1 \le i \le k-1, & & \\
(a_i, x_0), & 1 \le i \le k-1, & & \\
(a_i, x_1), & 1 \le i \le k-1, & & \\
(x_{D'}, b_i), & 1 \le i \le k-1, & & \\
(x_{D'-1}, b_i), & 1 \le i \le k-1, & & \\
(x_i, a_j), & 0 \le i \le D', & 1 \le j \le k-1, & \\
(b_j, x_i), & 0 \le i \le D', & 1 \le j \le k-1, & \\
(x_i, x_j), & 0 \le j < i \le D'. & &
\end{array}
$$

G is $(k+1)$-arc-connected, $|V(G)| = n = D' + 1 + 2(k-1)$. Let

$$E = \{(x_0, x_{D'}) \cup \{(a_i, b_i)\} \mid 1 \le i \le k-1\}.$$

Then $D(G-E) = D' = n - 2k + 1$. □

Theorem. *Let G be any $(k+1)$-arc-connected digraph of diameter 2. Let G' be the graph obtained by deleting k arcs of G. Then the diameter of G' is at most 4 and this bound is attained.*

Proof. The proof is quite exactly the same as in the undirected case.

Examples of graphs achieving the bound may be obtained from undirected examples by biorienting the edges.

5. Deletion of vertices

In [1] it is shown that if G is a λ-connected graph of order n and G' is the graph obtained from G by deleting t vertices, then $D(G') \le \lceil (n-t-2)/(\lambda - t) \rceil + 1$ and that the bound can be attained.

A proof exactly similar gives the same result for directed graphs.

Reference

[1] F.R.K. Chung and M.R. Garey, Diameter bounds for altered graphs, to appear.